Final Exam Solution

Problem 1

(1) The probability of exactly two voice calls is

$$P[N_V = 2] = P[\{vvd, vdv, dvv\}] = 0.3 \tag{1}$$

(2) The probability of at least one voice call is

$$P[N_V \geq 1] = P[\{vdd, dvd, ddv, vvd, vdv, dvv, vvv\}] \tag{2}$$

$$= 6(0.1) + 0.2 = 0.8 \tag{3}$$

An easier way to get the same answer is to observe that

$$P[N_V \geq 1] = 1 - P[N_V < 1] = 1 - P[N_V = 0] = 1 - P[\{ddd\}] = 0.8 \tag{4}$$

(3) The conditional probability of two voice calls followed by a data call given that there were two voice calls is

$$P[\{vvd\} | N_V = 2] = \frac{P[\{vvd\}, N_V = 2]}{P[N_V = 2]} = \frac{P[\{vvd\}]}{P[N_V = 2]} = \frac{0.1}{0.3} = \frac{1}{3} \tag{5}$$

(4) The conditional probability of two data calls followed by a voice call given there were two voice calls is

$$P[\{ddv\} | N_V = 2] = \frac{P[\{ddv\}, N_V = 2]}{P[N_V = 2]} = 0 \tag{6}$$

The joint event of the outcome ddv and exactly two voice calls has probability zero since there is only one voice call in the outcome ddv.

(5) The conditional probability of exactly two voice calls given at least one voice call is

$$P[N_V = 2 | N_v \geq 1] = \frac{P[N_V = 2, N_V \geq 1]}{P[N_V \geq 1]} = \frac{P[N_V = 2]}{P[N_V \geq 1]} = \frac{0.3}{0.8} = \frac{3}{8} \tag{7}$$

(6) The conditional probability of at least one voice call given there were exactly two voice calls is

$$P[N_V \geq 1 | N_V = 2] = \frac{P[N_V \geq 1, N_V = 2]}{P[N_V = 2]} = \frac{P[N_V = 2]}{P[N_V = 2]} = 1 \tag{8}$$

Given that there were two voice calls, there must have been at least one voice call.

Problem 2

The PMF of K is

$$P_K(k) = \begin{cases} 5^k e^{-5}/k! & k = 0, 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases} \tag{2.32}$$

Therefore $P[K = 0] = P_K(0) = e^{-5} = 0.0067$. To answer the question about the 2-second interval, we note in the problem definition that $\alpha = 5$ queries $= \lambda T$ with $T = 10$ seconds. Therefore, $\lambda = 0.5$ queries per second. If N is the number of queries processed in a 2-second interval, $\alpha = 2\lambda = 1$ and N is the Poisson (1) random variable with PMF

$$P_N(n) = \begin{cases} e^{-1}/n! & n = 0, 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases} \tag{2.33}$$

Therefore,

$$P[N \geq 2] = 1 - P_N(0) - P_N(1) = 1 - e^{-1} - e^{-1} = 0.264. \tag{2.34}$$

Problem 3

(a) We choose c so that the PMF sums to one.

$$\sum_x P_X(x) = \frac{c}{2} + \frac{c}{4} + \frac{c}{8} = \frac{7c}{8} = 1 \tag{1}$$

Thus $c = 8/7$.

(b)

$$P[X = 4] = P_X(4) = \frac{8}{7 \cdot 4} = \frac{2}{7} \tag{2}$$

(c)

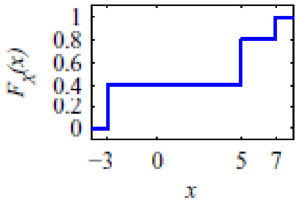
$$P[X < 4] = P_X(2) = \frac{8}{7 \cdot 2} = \frac{4}{7} \tag{3}$$

(d)

$$P[3 \leq X \leq 9] = P_X(4) + P_X(8) = \frac{8}{7 \cdot 4} + \frac{8}{7 \cdot 8} = \frac{3}{7} \tag{4}$$

Problem 4

(a) Similar to the previous problem, the graph of the CDF is shown below.



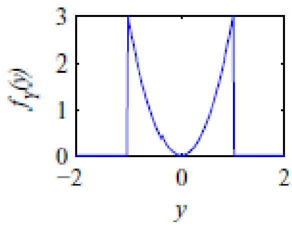
$$F_X(x) = \begin{cases} 0 & x < -3 \\ 0.4 & -3 \le x < 5 \\ 0.8 & 5 \le x < 7 \\ 1 & x \ge 7 \end{cases} \qquad (1)$$

(b) The corresponding PMF of X is

$$P_X(x) = \begin{cases} 0.4 & x = -3 \\ 0.4 & x = 5 \\ 0.2 & x = 7 \\ 0 & \text{otherwise} \end{cases} \qquad (2)$$

Problem 5

The PDF of Y is



$$f_Y(y) = \begin{cases} 3y^2/2 & -1 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

(1) The expected value of Y is

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\, dy = \int_{-1}^{1} (3/2) y^3\, dy = (3/8) y^4 \Big|_{-1}^{1} = 0. \quad (2)$$

Note that the above calculation wasn't really necessary because $E[Y] = 0$ whenever the PDF $f_Y(y)$ is an even function (i.e., $f_Y(y) = f_Y(-y)$).

(2) The second moment of Y is

$$E\left[Y^2\right] = \int_{-\infty}^{\infty} y^2 f_Y(y)\, dy = \int_{-1}^{1} (3/2) y^4\, dy = (3/10) y^5 \Big|_{-1}^{1} = 3/5. \quad (3)$$

(3) The variance of Y is

$$\text{Var}[Y] = E\left[Y^2\right] - (E[Y])^2 = 3/5. \quad (4)$$

(4) The standard deviation of Y is $\sigma_Y = \sqrt{\text{Var}[Y]} = \sqrt{3/5}$.

Problem 6

In this problem, the CDF of W is

$$F_W(w) = \begin{cases} 0 & w < \quad 5 \\ (w+5)/8 & -5 \le w < -3 \\ 1/4 & -3 \le w < 3 \\ 1/4 + 3(w-3)/8 & 3 \le w < 5 \\ 1 & w \ge 5. \end{cases} \tag{1}$$

Each question can be answered directly from this CDF.

(a)

$$P[W \le 4] = F_W(4) = 1/4 + 3/8 = 5/8. \tag{2}$$

(b)

$$P[-2 < W \le 2] = F_W(2) - F_W(-2) = 1/4 - 1/4 = 0. \tag{3}$$

(c)

$$P[W > 0] = 1 - P[W \le 0] = 1 - F_W(0) = 3/4 \tag{4}$$

(d) By inspection of $F_W(w)$, we observe that $P[W \le a] = F_W(a) = 1/2$ for a in the range $3 \le a \le 5$. In this range,

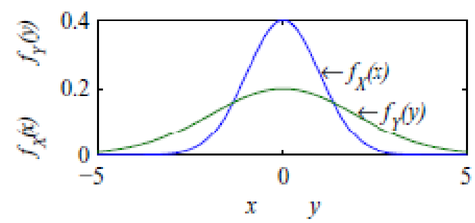
$$F_W(a) = 1/4 + 3(a-3)/8 = 1/2 \tag{5}$$

This implies $a = 11/3$.

Problem 7

Each of the requested probabilities can be calculated using $\Phi(z)$ function and Table 3.1 or $Q(z)$ and Table 3.2. We start with the sketches.

(1) The PDFs of X and Y are shown below. The fact that Y has twice the standard deviation of X is reflected in the greater spread of $f_Y(y)$. However, it is important to remember that as the standard deviation increases, the peak value of the Gaussian PDF goes down.



(2) Since X is Gaussian (0, 1),

$$P[-1 < X \leq 1] = F_X(1) - F_X(-1) \tag{1}$$
$$= \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 = 0.6826. \tag{2}$$

(3) Since Y is Gaussian (0, 2),

$$P[-1 < Y \leq 1] = F_Y(1) - F_Y(-1) \tag{3}$$
$$= \Phi\left(\frac{1}{\sigma_Y}\right) - \Phi\left(\frac{-1}{\sigma_Y}\right) = 2\Phi\left(\frac{1}{2}\right) - 1 = 0.383. \tag{4}$$

(4) Again, since X is Gaussian (0, 1), $P[X > 3.5] = Q(3.5) = 2.33 \times 10^{-4}$.

(5) Since Y is Gaussian (0, 2), $P[Y > 3.5] = Q(\frac{3.5}{2}) = Q(1.75) = 1 - \Phi(1.75) = 0.0401$.

Problem 8

From the joint PMF of Q and G given in the table, we can calculate the requested probabilities by summing the PMF over those values of Q and G that correspond to the event.

(1) The probability that $Q = 0$ is

$$P[Q=0] = P_{Q,G}(0,0) + P_{Q,G}(0,1) + P_{Q,G}(0,2) + P_{Q,G}(0,3) \quad (1)$$
$$= 0.06 + 0.18 + 0.24 + 0.12 = 0.6 \quad (2)$$

(2) The probability that $Q = G$ is

$$P[Q=G] = P_{Q,G}(0,0) + P_{Q,G}(1,1) = 0.18 \quad (3)$$

(3) The probability that $G > 1$ is

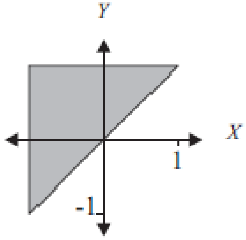
$$P[G>1] = \sum_{g=2}^{3}\sum_{q=0}^{1} P_{Q,G}(q,g) \quad (4)$$
$$= 0.24 + 0.16 + 0.12 + 0.08 = 0.6 \quad (5)$$

(4) The probability that $G > Q$ is

$$P[G>Q] = \sum_{q=0}^{1}\sum_{g=q+1}^{3} P_{Q,G}(q,g) \quad (6)$$
$$= 0.18 + 0.24 + 0.12 + 0.16 + 0.08 = 0.78 \quad (7)$$

Problem 9

(a) The joint PDF (and the corresponding region of nonzero probability) are



$$f_{X,Y}(x,y) = \begin{cases} 1/2 & -1 \le x \le y \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(b)

$$P[X > 0] = \int_0^1 \int_x^1 \frac{1}{2}\, dy\, dx = \int_0^1 \frac{1-x}{2}\, dx = 1/4 \tag{2}$$

This result can be deduced by geometry. The shaded triangle of the X, Y plane corresponding to the event $X > 0$ is 1/4 of the total shaded area.

(c) For $x > 1$ or $x < -1$, $f_X(x) = 0$. For $-1 \le x \le 1$,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy = \int_x^1 \frac{1}{2}\, dy = (1-x)/2. \tag{3}$$

The complete expression for the marginal PDF is

$$f_X(x) = \begin{cases} (1-x)/2 & -1 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

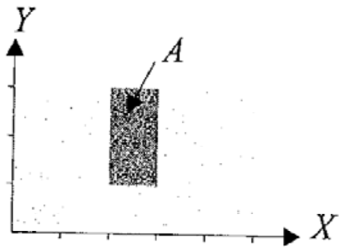
(d) From the marginal PDF $f_X(x)$, the expected value of X is

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \frac{1}{2} \int_{-1}^{1} x(1-x)\, dx \tag{5}$$

$$= \frac{x^2}{4} - \frac{x^3}{6} \bigg|_{-1}^{1} = -\frac{1}{3}. \tag{6}$$

Problem 10

The large rectangle in the diagram is the area of nonzero probability. Theorem 4.6 states that the integral of the joint PDF over this rectangle is 1:



$$1 = \int_0^5 \int_0^3 c\, dy\, dx = 15c. \tag{4.23}$$

Therefore, $c = 1/15$. The small dark rectangle in the diagram is the event $A = \{2 \le X < 3, 1 \le Y < 3\}$. $P[A]$ is the integral of the PDF over this rectangle, which is

$$P[A] = \int_2^3 \int_1^3 \frac{1}{15}\, dv\, du = 2/15. \tag{4.24}$$

This probability model is an example of a pair of random variables uniformly distributed over a rectangle in the X, Y plane.

Extra credit problem

In this problem, we use Theorem 3.14 and the tables for the Φ and Q functions to answer the questions. Since $E[Y_{20}] = 40(20) = 800$ and $\text{Var}[Y_{20}] = 100(20) = 2000$, we can write

$$P[Y_{20} > 1000] = P\left[\frac{Y_{20} - 800}{\sqrt{2000}} > \frac{1000 - 800}{\sqrt{2000}}\right] \tag{1}$$

$$= P\left[Z > \frac{200}{20\sqrt{5}}\right] = Q(4.47) = 3.91 \times 10^{-6} \tag{2}$$

The second part is a little trickier. Since $E[Y_{25}] = 1000$, we know that the prof will spend around \$1000 in roughly 25 years. However, to be certain with probability 0.99 that the prof spends \$1000 will require more than 25 years. In particular, we know that

$$P[Y_n > 1000] = P\left[\frac{Y_n - 40n}{\sqrt{100n}} > \frac{1000 - 40n}{\sqrt{100n}}\right] = 1 - \Phi\left(\frac{100 - 4n}{\sqrt{n}}\right) = 0.99 \tag{3}$$

Hence, we must find n such that

$$\Phi\left(\frac{100 - 4n}{\sqrt{n}}\right) = 0.01 \tag{4}$$

Recall that $\Phi(x) = 0.01$ for a negative value of x. This is consistent with our earlier observation that we would need $n > 25$ corresponding to $100 - 4n < 0$. Thus, we use the identity $\Phi(x) = 1 - \Phi(-x)$ to write

$$\Phi\left(\frac{100 - 4n}{\sqrt{n}}\right) = 1 - \Phi\left(\frac{4n - 100}{\sqrt{n}}\right) = 0.01 \tag{5}$$

Equivalently, we have

$$\Phi\left(\frac{4n - 100}{\sqrt{n}}\right) = 0.99 \tag{6}$$

From the table of the Φ function, we have that $(4n - 100)/\sqrt{n} = 2.33$, or

$$(n - 25)^2 = (0.58)^2 n = 0.3393n. \tag{7}$$

Solving this quadratic yields $n = 28.09$. Hence, only after 28 years are we 99 percent sure that the prof will have spent \$1000. Note that a second root of the quadratic yields $n = 22.25$. This root is not a valid solution to our problem. Mathematically, it is a solution of our quadratic in which we choose the negative root of $\sqrt{n}$. This would correspond to assuming the standard deviation of Y_n is negative.